

# The Fiction of Full BEKK: Pricing Fossil Fuels and Carbon Emissions

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## Abstract

The purpose of the paper is to (i) show that univariate GARCH is not a special case of multivariate GARCH, specifically the Full BEKK model, except under parametric restrictions on the off-diagonal elements of the random coefficient autoregressive coefficient matrix, that are not consistent with Full BEKK, and (ii) provide the regularity conditions that arise from the underlying random coefficient autoregressive process, for which the (quasi-) maximum likelihood estimates (QMLE) have valid asymptotic properties under the appropriate parametric restrictions. The paper provides a discussion of the stochastic processes that lead to the alternative specifications, regularity conditions, and asymptotic properties of the univariate and multivariate GARCH models. It is shown that the Full BEKK model, which in empirical practice is estimated almost exclusively compared with Diagonal BEKK (DBEKK), has no underlying stochastic process that leads to its specification, regularity conditions, or asymptotic properties, as compared with DBEKK. An empirical illustration shows the differences in the QMLE of the parameters of the conditional means and conditional variances for the univariate, DBEKK and Full BEKK specifications

## Keywords

Random coefficient stochastic process, Off-diagonal parametric restrictions, Diagonal BEKK, Full BEKK, Regularity conditions, Asymptotic properties, Conditional volatility, Univariate and multivariate models, Fossil fuels and carbon emissions.

## JEL Classification

C22, C32, C52, C58.

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## **Abstract**

The purpose of the paper is to (i) show that univariate GARCH is not a special case of multivariate GARCH, specifically the Full BEKK model, except under parametric restrictions on the off-diagonal elements of the random coefficient autoregressive coefficient matrix, that are not consistent with Full BEKK, and (ii) provide the regularity conditions that arise from the underlying random coefficient autoregressive process, for which the (quasi-) maximum likelihood estimates (QMLE) have valid asymptotic properties under the appropriate parametric restrictions. The paper provides a discussion of the stochastic processes that lead to the alternative specifications, regularity conditions, and asymptotic properties of the univariate and multivariate GARCH models. It is shown that the Full BEKK model, which in empirical practice is estimated almost exclusively compared with Diagonal BEKK (DBEKK), has no underlying stochastic process that leads to its specification, regularity conditions, or asymptotic properties, as compared with DBEKK. An empirical illustration shows the differences in the QMLE of the parameters of the conditional means and conditional variances for the univariate, DEBEKK and Full BEKK specifications.

**Keywords:** Random coefficient stochastic process, Off-diagonal parametric restrictions, Diagonal BEKK, Full BEKK, Regularity conditions, Asymptotic properties, Conditional volatility, Univariate and multivariate models, Fossil fuels and carbon emissions.

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# 1. Introduction

The most widely estimated univariate and multivariate models of time-varying volatility for financial data, as well as any high frequency data that are measured in days, hours and minutes, is the conditional volatility model. The underlying stochastic processes that lead to the specifications, regularity conditions and asymptotic properties of the most popular univariate conditional volatility models, such as GARCH (see Engle (1982) and Bollerslev (1986)) and GJR (see Glosten et al. (1993)) are well established in the literature, though McAleer and Hafner (2014) have raised caveats regarding the existence of the stochastic process underlying exponential GARCH (EGARCH) (see Nelson (1990, 1991)).

However, the same cannot be said about multivariate conditional volatility models, specifically Full BEKK (see Baba et al. (1985) and Engle and Kroner (1995)), for which the underlying stochastic process that leads to the specification, regularity conditions and asymptotic properties have either not been established, or are simply assumed rather than derived. These conditions are essential for forecasting and valid statistical analysis of the empirical estimates, which are the primary purposes of the models.

The purpose of the paper is to show that the stochastic process underlying univariate GARCH is not a special case of that underlying multivariate GARCH, except under parametric restrictions on the off-diagonal elements of the random coefficient autoregressive coefficient matrix that are not consistent with Full BEKK. The paper provides the regularity conditions that arise from the underlying random coefficient autoregressive process, and for which the (quasi-) maximum likelihood estimates (QMLE) have valid asymptotic properties under the appropriate parametric restrictions.

The Full BEKK model is estimated almost exclusively in empirical practice, to the exclusion of Diagonal BEKK (DBEKK), despite the fact that Full BEKK has no underlying stochastic process that leads to its specification, regularity conditions, or asymptotic properties, as shown in the proposition and four corollaries, as compared with DBEKK.

The plan of the paper is as follows. Section 2 provides a discussion of the stochastic processes,

regularity conditions, and asymptotic properties of univariate and multivariate GARCH models. Section 3 shows that the Full BEKK model has no underlying stochastic process that leads to its specification, regularity conditions, or asymptotic properties, as compared with DBEKK. In Section 4, an empirical illustration for the financial returns on spot and futures prices of fossil fuels and carbon emissions for the European Union and USA shows the differences that can arise in the QMLE of the parameters of the conditional means and conditional variances for the univariate, DBEKK and Full BEKK specifications. Section 5 gives some concluding comments.

## 2. Univariate and Multivariate GARCH Models

### 2.1 Univariate Conditional Volatility Models

Consider the conditional mean of financial returns for commodity  $i$ , in a financial portfolio of  $m$  assets, as follows:

$$y_{it} = E(y_{it}|I_{t-1}) + \varepsilon_{it}, \quad i = 1, 2, \dots, m, \quad (1)$$

where the returns,  $y_{it} = \Delta \log P_{it}$ , represent the log-difference in financial commodity prices,  $P_t, I_{t-1}$  is the information set for all financial assets at time  $t-1$ ,  $E(y_{it}|I_{t-1})$  is the conditional expectation of returns, and  $\varepsilon_{it}$  is a conditionally heteroskedastic error term.

In order to derive conditional volatility specifications, it is necessary to specify the stochastic processes underlying the returns shocks,  $\varepsilon_{it}$ . The most popular univariate conditional volatility model, GARCH model, is discussed below.

Consider the random coefficient autoregressive process underlying the returns shocks,  $\varepsilon_{it}$ , as follows:

$$\varepsilon_{it} = \phi_{it}\varepsilon_{it-1} + \eta_{it}, \quad i = 1, 2, \dots, m, \quad (2)$$

where

$\phi_{it} \sim iid(0, \alpha_i), \alpha_i \geq 0,$   
 $\eta_{it} \sim iid(0, \omega_i), \omega_i \geq 0,$   
 $\eta_{it} = \varepsilon_{it} / \sqrt{h_{it}}$  is the standardized residual,  
 $h_{it}$  is the conditional volatility of financial asset  $i$ .

Tsay (1987) derived the following conditional volatility of financial asset  $i$  as an ARCH process (see Engle, 1982):

$$E(\varepsilon_{it}^2 | I_{t-1}) \equiv h_{it} = \omega_i + \alpha_i \varepsilon_{it-1}^2, \quad (3)$$

where  $h_t$  represents conditional volatility, and  $I_{t-1}$  is the information set available at time  $t-1$ . A lagged dependent variable,  $h_{t-1}$ , is typically added to equation (3) to improve the sample fit:

$$h_{it} \equiv E(\varepsilon_{it}^2 | I_{t-1}) = \omega_i + \alpha_i \varepsilon_{it-1}^2 + \beta_i h_{t-1}, \quad \beta_i \in (-1, 1). \quad (4)$$

From the specification of equation (2), it is clear that both  $\omega_i$  and  $\alpha_i$  should be positive as they are the unconditional variances of two different stochastic processes. In equation (4), which is a GARCH(1,1) model for commodity  $i$  (see Bollerslev, 1986), the stability condition requires that  $\beta_i \in (-1, 1)$ .

The stochastic process can be extended to asymmetric conditional volatility models (see, for example, McAleer (2014)), and to give higher-order lags and a larger number of alternative commodities, namely up to  $m-1$ . However, the symmetric process considered here is sufficient to focus the key ideas associated with the purpose of the paper.

As the stochastic process in equation (2) follows a random coefficient autoregressive process, under normality (non-normality) of the random errors, the maximum likelihood estimators (quasi- maximum likelihood estimators, QMLE) of the parameters will be consistent and asymptotically normal. It is worth emphasizing that the regularity conditions include invertibility, which is obvious from equation (2), as:

$$\varepsilon_{it} - \phi_{it}\varepsilon_{it-1} = \eta_{it}.$$

The standardized residuals,  $\eta_{it}$ , can be expressed in terms of the empirical data through equations (1) and (2), as  $\varepsilon_{it}$  can be estimated using equation (1),  $\varepsilon_{it-1}$  is the lagged value, which has already been estimated, and the random coefficient can be generated under appropriate explicit assumptions regarding its underlying stochastic process. In short,  $\eta_{it}$  can be related directly to the data,  $y_{it}$ , using equations (1) and (2).

Ling and McAleer (2003) and McAleer et al. (2008) provide general proofs of the asymptotic properties of univariate and multivariate conditional volatility models based on satisfying the regularity conditions in Jeantheau (1998) for consistency, and in Theorem 4.1.3 in Amemiya (1985) for asymptotic normality.

## 2.2 Multivariate Conditional Volatility Models

The multivariate extension of the univariate ARCH and GARCH models is given in Baba et al. (1985) and Engle and Kroner (1995). It is useful to define the multivariate extension of the relationship between the returns shocks and the standardized residuals, that is,  $\eta_{it} = \varepsilon_{it}/\sqrt{h_{it}}$ . The multivariate extension of equation (1), namely:

$$y_t = E(y_t|I_{t-1}) + \varepsilon_t, \quad (5)$$

can remain unchanged by assuming that each of the three components in equation (5) is an  $m \times 1$  vector, where  $m$  is the number of financial assets.

The following two definitions are intended to elaborate on the discussion below:

**Definition 1:** Each marginal of  $\varepsilon_{it}$  should be a univariate counterpart of the multivariate returns vector,  $\varepsilon_t$ .

**Definition 2:** An underlying stochastic process of a univariate returns shock, or multivariate

returns shocks, is one that leads to the regularity conditions, likelihood function, and asymptotic properties of the resulting quasi- maximum likelihood estimators.

Consider the vector random coefficient autoregressive process of order one, which is the multivariate extension of the univariate process given in equation (2):

$$\varepsilon_t = \Phi_t \varepsilon_{t-1} + \eta_t, \quad (6)$$

where

$\varepsilon_t$  and  $\eta_t$  are  $m \times 1$  vectors,

$\Phi_t$  is an  $m \times m$  matrix of random coefficients,

$\Phi_t \sim iid(0, A)$ ,  $A$  is positive definite,

$\eta_t \sim iid(0, C)$ ,  $C$  is an  $m \times m$  matrix.

Vectorization of a full matrix  $A$  to  $vec A$  can have dimension as high as  $m^2 \times m^2$ , whereas vectorization of a symmetric matrix  $A$  to  $vech A$  can have a smaller dimension of  $m(m+1)/2 \times m(m+1)/2$ .

In the case where  $A$  is a diagonal matrix, with  $a_{ii} > 0$  for all  $i = 1, \dots, m$  and  $|b_{jj}| < 1$  for all  $j = 1, \dots, m$ , so that  $A$  has dimension  $m \times m$ , McAleer et al. (2008) showed that the multivariate extension of GARCH(1,1) from equation (6) is given as the Diagonal BEKK (DBEKK) model, namely:

$$Q_t = CC' + A\varepsilon_{t-1}\varepsilon_{t-1}'A' + BQ_{t-1}B', \quad (7)$$

where  $A$  and  $B$  are both diagonal matrices. The diagonality of the positive definite matrix  $A$  is essential for matrix multiplication as  $\varepsilon_{t-1}\varepsilon_{t-1}'$  is an  $m \times m$  matrix; otherwise equation (7) could not be derived from the vector random coefficient autoregressive process in equation (6).

McAleer et al. (2008) showed that the QMLE of the parameters of the DBEKK model were consistent and asymptotically normal, so that standard statistical inference on testing hypotheses



is valid (or further details, see Chang et al., 2018). It should be emphasized that the QMLE of the parameters in the conditional means, namely equations (1) and (5), and the conditional variances, namely equations (4) and (7), will differ as the multivariate models, (5) and (7), respectively, are estimated jointly, whereas the univariate models, (1) and (4), respectively, are estimated individually.

### 3. Full BEKK

Consider element  $i$  of equation (6), that is:

$$\varepsilon_{it} = \sum_{j=1}^m \phi_{ijt} \varepsilon_{ijt-1} + \eta_{it}, \quad i = 1, 2, \dots, m, \quad (8)$$

which is not equivalent to equation (2) unless  $\phi_{ijt} = 0 \quad \forall i \neq j$ . Such parametric restrictions are not consistent with the Full BEKK specification, which assumes  $\phi_{ijt} \neq 0$  for at least one  $i \neq j$ ,  $i, j = 1, 2, \dots, m$ .

The stochastic process given in equation (8) is not a random coefficient autoregressive process because of the presence of an additional  $m-1$  random coefficients,  $\phi_{ijt}$ ,  $i \neq j$ . Importantly, equation (8) is **not** invertible as the standardized residual,  $\eta_{it}$ , cannot be connected to the data,  $y_{it}$ , as  $m$  equations are required, as in equation (6). Consequently, the stochastic process underlying univariate ARCH is not a special case of the stochastic process underlying multivariate ARCH unless  $\phi_{ijt} = 0 \quad \forall i \neq j$ .

The same condition holds  $\forall i, j = 1, \dots, m$ , which leads to the following proposition:

**Proposition:** The stochastic process underlying univariate ARCH in equation (2) is a special case of the stochastic process underlying multivariate ARCH in equation (8) if and only if:

$$\phi_{ijt} = 0 \quad \forall i \neq j, i, j = 1, 2, \dots, m.$$

**Proof:** If  $\phi_{ijt} = 0 \ \forall i \neq j$ , equation (8) collapses to equation (2), with  $\phi_{iit} = \phi_{it}$ . If  $\phi_{ijt} \neq 0$  for at least one  $i \neq j$ , equation (2) is not a special case of equation (8).

A similar condition holds for univariate GARCH and multivariate GARCH.

The Proposition leads to the following corollaries:

**Corollary 1:** The  $m \times m$  matrix of random coefficients,  $\Phi_t$ , is a diagonal matrix.

**Corollary 2:** From Corollary 1, it follows that the  $m \times m$  weight matrix of (co-)variances,  $A$ , is a diagonal matrix, which is not consistent with Full BEKK.

**Corollary 3:** Corollaries 1 and 2 show that a Full BEKK model, namely where there are no restrictions on the off-diagonal elements in  $\Phi_t$ , and hence no restrictions in the off-diagonal elements in  $A$ , is not possible if univariate ARCH is to be a special case of its multivariate counterpart, Full BEKK.

**Corollary 4:** As there are no underlying regularity conditions for Full BEKK, including invertibility, the model cannot be estimated using an appropriate likelihood function. Therefore, it is not possible to derive the asymptotic properties of the QMLE of the unknown parameters in the Full BEKK specification.

Corollary 4 is consistent with the proof in McAleer et al. (2008) that the QMLE of Full BEKK has no asymptotic properties, whereas the QMLE of Diagonal BEKK can be shown to be consistent and asymptotically normal.

For all intents and purposes, the statistical properties of Full BEKK cannot be derived from an underlying stochastic process, except by assumption.

It should be emphasized that the QMLE of the parameters in the conditional means and the conditional variances for univariate GARCH, DBEKK and Full BEKK will differ as the

multivariate models are estimated jointly, whereas the univariate models are estimated individually. The QMLE of the parameters of the conditional means and the conditional variances of DBEKK and Full BEKK will differ as DBEKK imposes parametric restrictions on the off-diagonal terms of the conditional covariance matrix of Full BEKK.

#### **4. An Empirical Illustration for Fossil Fuels and Carbon Emissions**

The data for the empirical analysis are given in Chang et al. (2017), who evaluated the financial returns on spot and futures prices for fossil fuels and carbon emissions for the European Union and USA using the DBEKK and Full BEKK models. The authors did not provide the estimates for the univariate GARCH models, or compare the differences in the conditional means and conditional variances of the univariate, DBEKK and Full BEKK specifications. The purpose of the empirical illustration in this section is to show the differences that can arise in the QMLE of the parameters of the conditional means and conditional variances of the univariate, DBEKK and Full BEKK specifications.

The carbon emission trading market of the European Union (EU) has daily data only on futures prices, whereas only daily spot prices are available for carbon emissions for the USA. Daily data for EU carbon emission, crude oil, and coal futures are available from 2 April 2008 to 19 May 2017, while daily data for US carbon, coal, and oil spot prices are available from 6 January 2016 to 19 May 2017. The data sources and definitions are given in Table 1, where “fr” denotes futures returns, “sr” denotes spot returns, and daily returns are calculated as obtained as the first difference in the natural logarithm of the relevant daily price data.

The descriptive statistics for the returns of the six variables are given in Table 2 (for a detailed discussion of the data, see Chang et al., 2017). Table 3 presents the ADF test of Dickey and Fuller (1979, 1982) and Said and Dickey (1984), the DF-GLS test of Elliott et al. (1996), and the KPSS test of Kwiatkowski et al. (1992) to test for unit roots in the individual returns series (see Chang et al., 2017).

The univariate GARCH estimates for EU carbon, coal and oil futures returns are given in Table 4. The QMLE of the parameters of the conditional means are standard in that there is not a lot of explanatory power. However, the QMLE of the parameters of the conditional variances are highly significant, with the short run responses to shocks being around 0.1 or less, and the long run responses to shocks lying between 0.996 and 0.997.

The univariate GARCH estimates for US carbon, coal and oil spot returns are given in Table 5. The QMLE of the parameters of the conditional means are similar to those in Table 4 in that there is not a lot of explanatory power. However, the QMLE of the parameters of the conditional variances are highly significant. The short run responses to shocks are surprisingly large for carbon at 0.462, while those for coal and oil are more standard at 0.073 and 0.130, respectively. Give these estimates, the long run responses to shocks are 0.936, 0.982 and 0.954 for carbon, coal and oil, respectively, all of which are considerably lower than their counterparts for EU futures returns.

The corresponding estimates for the DBEKK and Full BEKK models for EU carbon, coal and oil futures returns are given in Tables 6 and 7, respectively. The QMLE of the conditional means for DBEKK and Full BEKK are different from each other, and are also different from their univariate counterparts in Table 4. The QMLE of the elements of the weighting matrix A and stability matrix B, namely  $a_{11}$ ,  $a_{22}$ ,  $a_{33}$ ,  $b_{11}$ ,  $b_{22}$  and  $b_{33}$ , respectively, are substantially different between both DBEKK (especially  $a_{22}$  and  $b_{33}$ ) and Full BEKK (especially  $a_{22}$ ,  $a_{33}$  and  $b_{33}$ ), and even more so in comparison with their univariate counterparts in Table 4. These results provide strong support for the theoretical analysis in Sections 2 and 3.

The corresponding estimates for the DBEKK and Full BEKK models for US carbon, coal and oil spot returns are given in Tables 8 and 9, respectively. The QMLE of the conditional means for DBEKK and Full BEKK are different from each other, and are also different from their univariate counterparts in Table 5. The QMLE of the elements of the weighting matrix A and stability matrix B, namely  $a_{11}$ ,  $a_{22}$ ,  $a_{33}$ ,  $b_{11}$ ,  $b_{22}$  and  $b_{33}$ , respectively, are substantially different between both DBEKK (especially  $a_{22}$ ,  $a_{33}$  and  $b_{33}$ ) and Full BEKK (especially  $a_{22}$ ,  $a_{33}$  and  $b_{33}$ ), which reflect the findings in Tables 6 and 7, and even more so in comparison with

their univariate counterparts in Table 4. These results also strongly support the theoretical analysis in Sections 2 and 3.

## 5. Conclusion

The Full BEKK model in Baba et al. (1985) and Engle and Kroner (1995), who do not derive the model from an underlying stochastic process, was presented as equation (6), with  $A$  and  $B$  given as full matrices, with no restrictions on the off-diagonal elements. The Full BEKK model is estimated almost exclusively in empirical practice, to the exclusion of Diagonal BEKK, despite the fact that Full BEKK has no underlying stochastic process that leads to its specification, regularity conditions, or asymptotic properties, as shown in the proposition and four corollaries.

The full BEKK model can be replaced by the triangular or Hadamard (element-by-element multiplication) BEKK models, with similar problems of identification and (lack of) existence. The full, triangular and Hadamard BEKK models cannot be derived from any known underlying stochastic processes that lead to their respective specifications, which means there are no regularity conditions (except by assumption) for checking the internal consistency of the alternative models, and consequently no valid asymptotic properties of the QMLE of the associated parameters (except by assumption).

Moreover, as the number of parameters in a full BEKK model can be as much as  $3m(m+1)/2$ , the “curse of dimensionality” will be likely to arise, which means that convergence of the estimation algorithm can become problematic and less reliable when there is a large number of parameters to be estimated. As a matter of fact, estimation of the full BEKK can be problematic even when  $m$  is as low as 5 financial assets. Such computational difficulties do not arise for the diagonal BEKK model. Convergence of the estimation algorithm is more likely when the number of commodities is less than 4, though this is nevertheless problematic in terms of interpretation.

The purpose of the paper was to show that univariate GARCH is not a special case of multivariate GARCH, specifically the Full BEKK model, except under parametric restrictions on a random coefficient autoregressive coefficient matrix that are not consistent with Full BEKK.

The paper provided the regularity conditions that arise from the underlying random coefficient autoregressive process, and for which the (quasi-) maximum likelihood estimates have valid asymptotic properties under the appropriate parametric restrictions, for the univariate and multivariate GARCH models.

It was shown that the Full BEKK model has no underlying stochastic process that leads to its specification, regularity conditions, or asymptotic properties, as compared with the Diagonal BEKK (DBEKK) specification. It would seem that the purported statistical properties of Full BEKK exist by assumption.

An empirical illustration for the financial returns on spot and futures prices of fossil fuels and carbon emissions for the European Union and USA showed the significant differences that can arise in the QMLE of the parameters of the conditional means and conditional variances for the univariate, DBEKK and Full BEKK specifications, which gave strong support for the theoretical analysis demonstrated in the paper.

**Table 1**  
**Data Sources and Definitions**

<b>Variable name</b>	<b>Definitions</b>	<b>Transaction market</b>	<b>Description</b>
<b>EUcarbon<sub>fr</sub></b>	EU carbon futures return	ICE-ICE Futures Europe Commodities	ICE EUA Futures Contract EUR/MT
<b>EUcoal<sub>fr</sub></b>	EU coal futures return	ICE-ICE Futures Europe Commodities	ICE Rotterdam Monthly Coal Futures Contract USD/MT
<b>EUoil<sub>fr</sub></b>	EU oil futures return	ICE-ICE Futures Europe Commodities	Current pipeline export quality Brent blend as supplied at Sullom Voe USD/bbl
<b>UScarbon<sub>sr</sub></b>	US carbon spot return	over the counter	United States Carbon Dioxide RGGI Allowance USD/Allowance
<b>UScoal<sub>sr</sub></b>	US coal spot return	over the counter	Dow Jones US Total Market Coal Index USD
<b>USoil<sub>sr</sub></b>	US oil spot return	over the counter	West Texas Intermediate Cushing Crude Oil USD/bbl

**Notes:** ICE is the Intercontinental Exchange; EUA is the EU allowance; MT is metric ton; RGGI (Regional Greenhouse Gas Initiative) is a CO<sub>2</sub> cap-and-trade emissions trading program comprised of ten New England and Mid-Atlantic States that will commence in 2009 and aims to reduce emissions from the power sector. RGGI will be the first government mandated CO<sub>2</sub> emissions trading program in USA.

**Table 2**  
**Descriptive Statistics**

**2 April 2008 – 19 May 2017 for EU**  
**6 January 2016 – 19 May 2017 for USA**

Variable	Mean	Median	Max	Min	SD	Skewness	Kurtosis	Jarque-Bera
<b>EUcarbon<sub>f</sub></b>	-0.078	-0.038	24.561	-42.457	3.349	-0.708	17.624	21434.2
<b>EUcoal<sub>fr</sub></b>	-0.022	0	17.419	-22.859	1.599	-1.268	44.924	175155.8
<b>EUoil<sub>fr</sub></b>	-0.026	-0.015	12.707	-10.946	2.246	0.054	6.522	1232.8
<b>UScarbon<sub>s</sub></b>	-0.248	0	13.937	-36.446	2.986	-5.236	66.269	61346.8
<b>UScoal<sub>sr</sub></b>	0.177	0.104	17.458	-14.183	4.041	0.047	5.343	81.99
<b>USoil<sub>sr</sub></b>	0.094	0.037	11.621	-8.763	2.712	0.431	4.690	53.69



**Table 3**  
**Unit Root Tests**

**2 April 2008 – 19 May 2017 for EU**  
**6 January 2016 – 19 May 2017 for USA**

<b>Variables</b>	<b>ADF</b>	<b>DF-GLS</b>	<b>KPSS</b>
<b>EUcarbon<sub>fr</sub></b>	-37.79*	-3.09*	0.05*
<b>EUcoal<sub>fr</sub></b>	-35.48*	-10.34*	0.12*
<b>EUoil<sub>fr</sub></b>	-51.97*	-1.53	0.10*
<b>UScarbon<sub>sr</sub></b>	-10.64*	-1.46	0.06*
<b>UScoal<sub>sr</sub></b>	-19.30*	-0.43	0.18*
<b>USoil<sub>sr</sub></b>	-20.96*	-0.78	0.07*

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**Notes:** \* denotes the null hypothesis of a unit root is rejected at 1%.

**Table 4****Univariate GARCH for EU CARBON<sub>fr</sub>, COAL<sub>fr</sub>, OIL<sub>fr</sub>****2 April 2008 – 19 May 2017**

<b>Explained variables</b>	<b>CARBON<sub>fr</sub></b> (1)	<b>COAL<sub>fr</sub></b> (2)	<b>OIL<sub>fr</sub></b> (3)
<b>Constant</b>	0.032 (0.050)	-0.040* (0.024)	0.003 (0.033)
$\theta_1$	0.017 (0.024)	0.097*** (0.023)	-0.039* (0.021)
$\theta_2$	-0.090** (0.040)	0.003 (0.007)	0.008 (0.008)
$\theta_3$	-0.055** (0.023)	0.010 (0.013)	-0.008 (0.028)
$\omega$	-0.116*** (0.037)	0.009*** (0.002)	0.020*** (0.007)
<b>GARCH <math>\alpha</math></b>	0.101*** (0.015)	0.016*** (0.002)	0.060*** (0.010)
<b>GARCH <math>\beta</math></b>	0.895*** (0.016)	0.980*** (0.002)	0.937*** (0.010)
<b>Log Likelihood</b>	-5874.33	-4030.45	-4872.13

**Notes:** (1) : CARBON<sub>fr</sub> = ( $\theta_1$  CARBON<sub>fr</sub>(-1),  $\theta_2$  COAL<sub>fr</sub>(-1),  $\theta_3$  OIL<sub>fr</sub>(-1))

(2): COAL<sub>fr</sub> = ( $\theta_1$  COAL<sub>fr</sub>(-1),  $\theta_2$  CARBON<sub>fr</sub>(-1),  $\theta_3$  OIL<sub>fr</sub>(-1))

(3): OIL<sub>fr</sub> = ( $\theta_1$  OIL<sub>fr</sub>(-1),  $\theta_2$  CARBON<sub>fr</sub>(-1),  $\theta_3$  COAL<sub>fr</sub>(-1))

Standard errors are in parentheses, \*\*\* denotes significant at 1%, \*\* denotes significant at 5%,  
\* denotes significant at 10%.

**Table 5**

**Univariate GARCH for US CARBON<sub>sr</sub>, COAL<sub>sr</sub>, OIL<sub>sr</sub>**

**6 January 2016 – 19 May 2017**

<b>Explained variables</b>	<b>CARBON<sub>sr</sub></b> (4)	<b>COAL<sub>sr</sub></b> (5)	<b>OIL<sub>sr</sub></b> (6)
<b>Constant</b>	0.049 (0.096)	0.029 (0.174)	0.074 (0.116)
$\theta_1$	0.100 (0.100)	0.020 (0.058)	-0.082 (0.060)
$\theta_2$	0.012 (0.025)	0.038 (0.078)	-0.097* (0.056)
$\theta_3$	-0.081** (0.038)	-0.238*** (0.080)	0.038 (0.038)
$\omega$	0.729*** (0.170)	0.211 (0.147)	0.274* (0.147)
<b>GARCH <math>\alpha</math></b>	0.462*** (0.091)	0.073** (0.030)	0.130** (0.044)
<b>GARCH <math>\beta</math></b>	0.574*** (0.052)	0.909*** (0.034)	0.824*** (0.055)
<b>Log Likelihood</b>	-759.38	-952.67	-816.74

**Notes:** (4) : CARBON<sub>sr</sub> = (  $\theta_1$  CARBON<sub>sr</sub>(-1),  $\theta_2$  COAL<sub>sr</sub>(-1),  $\theta_3$  OIL<sub>sr</sub>(-1))

(5): COAL<sub>sr</sub> = (  $\theta_1$  COAL<sub>sr</sub>(-1),  $\theta_2$  CARBON<sub>sr</sub>(-1),  $\theta_3$  OIL<sub>sr</sub>(-1))

(6): OIL<sub>sr</sub> = (  $\theta_1$  OIL<sub>sr</sub>(-1),  $\theta_2$  CARBON<sub>sr</sub>(-1),  $\theta_3$  COAL<sub>sr</sub>(-1))

Standard errors are in parentheses, \*\*\* denotes significant at 1%, \*\* denotes significant at 5%,  
\* denotes significant at 10%.

**Table 6**  
**DBEKK for EU Carbon, Coal, and Oil Futures**  
**2 April 2008 – 19 May 2017**

Mean equation	CARBON <sub>fr</sub>	COAL <sub>fr</sub>	OIL <sub>fr</sub>
CARBON <sub>fr</sub>	0.010 (0.023)	0.005 (0.008)	0.009 (0.009)
COAL <sub>fr</sub>	-0.078** (0.038)	0.096*** (0.023)	0.073 (0.023)
OIL <sub>fr</sub>	-0.057** (0.024)	0.009 (0.014)	0.002 (0.027)
C	0.021 (0.053)	-0.034 (0.024)	-0.045* (0.022)

DBEKK	C			A	B
CARBON <sub>fr</sub>	0.379*** (0.055)	0.024** (0.010)	0.128*** (0.024)	0.311*** (0.025)	0.947*** (0.009)
COAL <sub>fr</sub>		0.088*** (0.010)	0.022 (0.075)	0.118*** (0.007)	0.991*** (0.001)
OIL <sub>fr</sub>			0.000 (0.077)	-0.205*** (0.013)	-0.977*** (0.003)

**Notes:** 1.  $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ ,  $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$ ,  $C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$

2. Standard errors are in parentheses, \*\*\* denotes significant at 1%, \*\* denotes significant at 5%,  
\* denotes significant at 10%.

**Table 7**  
**Full BEKK for EU Carbon, Coal, and Oil Futures**  
**2 April 2008 – 19 May 2017**

Mean equation	CARBON <sub>fr</sub>	COAL <sub>fr</sub>	OIL <sub>fr</sub>
CARBON <sub>fr</sub>	0.023 (0.02)	-0.003 (0.007)	0.013 (0.011)
COAL <sub>fr</sub>	-0.082** (0.039)	0.086*** (0.023)	0.005 (0.031)
OIL <sub>fr</sub>	-0.045* (0.023)	0.016 (0.015)	-0.018 (0.023)
C	0.031 (0.053)	-0.016 (0.023)	-0.010 (0.037)

Full BEKK	C			A			B		
CARBON <sub>fr</sub>	0.435*** (0.055)	-0.067* (0.038)	0.077 (0.072)	0.331*** (0.023)	-0.014*** (0.004)	0.007 (0.006)	0.936*** (0.009)	0.009 (0.007)	-0.005 (0.010)
COAL <sub>fr</sub>		0.000 (0.068)	0.000 (0.103)	0.037 (0.029)	-0.086*** (0.011)	0.120*** (0.017)	0.274*** (0.036)	0.737*** (0.015)	1.110*** (0.023)
OIL <sub>fr</sub>			-0.000 (0.101)	-0.104*** (0.026)	-0.032** (0.013)	-0.168*** (0.010)	-0.189*** (0.024)	-0.052*** (0.011)	0.054*** (0.015)

Notes : As in Table 4.

**Table 8**  
**DBEKK for US Carbon, Coal, and Oil Spot**  
**6 January 2016 – 19 May 2017**

Mean equation	CARBON <sub>sr</sub>	COAL <sub>sr</sub>	OIL <sub>sr</sub>
CARBON <sub>sr</sub>	0.122 (0.106)	-0.010 (0.078)	-0.070 (0.053)
COAL <sub>sr</sub>	0.034 (0.024)	0.037 (0.057)	0.050 (0.041)
OIL <sub>sr</sub>	-0.097*** (0.036)	-0.235*** (0.083)	-0.103* (0.060)
C	0.085 (0.090)	0.048 (0.170)	0.010 (0.122)

DBEKK	C			A	B	
CARBON <sub>sr</sub>	0.854*** (0.105)	-0.276 (0.294)	0.129 (0.332)	0.707*** (0.073)	0.757*** (0.038)	
COAL <sub>sr</sub>		0.256 (0.314)	0.299* (0.154)	-0.199*** (0.034)	0.972*** (0.008)	
OIL <sub>sr</sub>			0.000 (1.029)	-0.222*** (0.0035)		-0.964*** (0.010)

**Note:** As in Table 4.

**Table 9**  
**Full BEKK for US Carbon, Coal, and Oil Spot**  
**6 January 2016 – 19 May 2017**

Mean equation	CARBON <sub>sr</sub>	COAL <sub>sr</sub>	OIL <sub>sr</sub>
CARBON <sub>sr</sub>	0.079 (0.089)	-0.027 (0.074)	-0.105** (0.049)
COAL <sub>sr</sub>	-0.006 (0.028)	-0.012 (0.060)	0.022 (0.039)
OIL <sub>sr</sub>	-0.048 (0.038)	-0.231*** (0.087)	-0.049 (0.062)
C	0.043 (0.089)	0.139 (0.166)	0.010 (0.118)

Full BEKK	C			A			B		
CARBON <sub>sr</sub>	0.772*** (0.092)	0.119 (0.606)	0.685*** (0.178)	0.632*** (0.054)	-0.023 (0.089)	-0.077 (0.064)	0.791*** (0.025)	0.004 (0.112)	-0.034 (0.063)
COAL <sub>sr</sub>		0.000 (0.528)	0.000 (0.715)	0.002 (0.033)	-0.320*** (0.058)	0.036 (0.041)	-0.042 (0.046)	0.900*** (0.056)	0.578*** (0.044)
OIL <sub>sr</sub>			0.000 (0.721)	-0.028 (0.049)	-0.072 (0.092)	-0.252*** (0.060)	0.010 (0.080)	-1.267*** (0.074)	0.140* (0.082)

**Note:** As in Table 4.

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